

**GLOBAL WELL-POSEDNESS AND LONG-TIME ASYMPTOTICS
FOR THE DEFOUSSING DAVEY-STEWARTSON II EQUATION
IN $H^{1,1}(\mathbb{R}^2)$**

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With an Appendix by Michael Christ

ABSTRACT. We use the $\bar{\partial}$ -inverse scattering method to obtain global well-posedness and large-time asymptotics for the defocussing Davey-Stewartson II equation. We show that these global solutions are dispersive by computing their leading asymptotic behavior as $t \rightarrow \infty$ in terms of an associated linear problem. These results appear to be sharp.

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1. INTRODUCTION

In this paper we will prove global well-posedness for the defocussing Davey-Stewartson II (DS II) equation

$$(1.1) \quad \begin{aligned} i q_t + 2 \left(\bar{\partial}^2 + \partial^2 \right) q + (g + \bar{g}) q &= 0, \\ \bar{\partial} g + \partial \left(|q|^2 \right) &= 0, \end{aligned}$$

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a nonlinear, completely integrable dispersive equation in two space dimensions. Here ∂ and $\bar{\partial}$ are the usual operators ($z = x + iy$)

$$\begin{aligned}\partial &= \frac{1}{2} (\partial_x - i\partial_y), \\ \bar{\partial} &= \frac{1}{2} (\partial_x + i\partial_y)\end{aligned}$$

and $q = q(z, \bar{z}, t)$. The DS II equation may be regarded as a two-dimensional analogue of the defocussing cubic nonlinear Schrödinger equation in one space dimension; it is one of a multiparameter family of models proposed by Benny-Roskes [12] and Davey-Stewartson [18] to model the propagation of nonlinear surface waves in shallow water (see Ghidaglia-Saut [21] for a physical derivation and for extensive local well-posedness results).

Here we will prove that the Cauchy problem for (1.1) is globally well-posed for initial data q_0 in the space

$$H^{1,1}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : \nabla f, (1 + |\cdot|) f(\cdot) \in L^2\}.$$

We will also show that the large-time asymptotics are determined by the linear problem

$$(1.2) \quad \begin{aligned}iu_t + 2(\bar{\partial}^2 + \partial^2)u &= 0, \\ u|_{t=0} &= u_0.\end{aligned}$$

Here $u_0 = \mathcal{F}^{-1}(\mathcal{R}q_0)$ where \mathcal{F} is essentially the Fourier transform (see (1.7)) and \mathcal{R} is a nonlinear scattering transform, defined below, associated to the DS II equation. Thus, in contrast to the cubic NLS in one dimension, there is no “logarithmic phase shift” due to the nonlinearity (see Deift and Zhou [19] for a thorough analysis of this phenomenon and references to the literature). Our proof will exploit the complete integrability of (1.1).

To formulate the main result, we recall how a solution of (1.1) is defined for non-smooth initial data. Following Ghidaglia and Saut [21], we say that a function $q \in C([0, T], L^2(\mathbb{R}^2)) \cap L^4([0, T] \times \mathbb{R}^2)$ solves the Davey-Stewartson II equation with initial data $q_0 \in L^2(\mathbb{R}^2)$ if $q(t)$ solves the integral equation

$$(1.3) \quad q(t) = S(t)q_0 - i \int_0^t S(t-s) \left(2q(s) \operatorname{Re} \left(\bar{\partial}^{-1} \partial \right) (|q(s)|^2) \right) ds,$$

for $t \in (0, T)$, where $S(t)$ is the solution operator for the linear Cauchy problem (1.2). Ghidaglia and Saut [21] proved, among other results, that the Davey-Stewartson II equation has solutions in this sense locally in time. We will prove:

Theorem 1.1. *Let $q_0 \in H^{1,1}(\mathbb{R}^2)$. There exists a locally Lipschitz continuous map*

$$\begin{aligned}H^{1,1}(\mathbb{R}^2) \times \mathbb{R} &\rightarrow H^{1,1}(\mathbb{R}^2) \\ (q_0, t) &\mapsto q(t)\end{aligned}$$

so that for each $q_0 \in \mathcal{S}(\mathbb{R}^2)$, the function q is a classical solution of the Davey-Stewartson II equation (1.1). Moreover, $\|q(t)\|_2$ is conserved.

Remark 1.2. Since $H^{1,1}(\mathbb{R}^2) \subset L^p(\mathbb{R}^2)$ for all $p \in (1, \infty)$, it is easy to see that the global solution so obtained coincides with the local Ghidaglia-Saut solution for all T , and thus shows that these solutions extend to $T = \infty$.

Remark 1.3. The same analysis used here can be used to show global existence for the focussing DS II equation that differs from (1.1) in the sign of the nonlinearity and sufficiently small initial data. Ozawa [25] constructed a solution to the focussing DS II equation with the following properties: (1) the initial data $q_0 \in L^2$, but $|\nabla q_0(z)|, |zq(z)| \geq C(1 + |z|)^{-1}$ for a positive constant C , (2) the measure $|q(z, t)|^2 dm(z)$ concentrates to a δ -function in finite time (see also C. Sulem and P. Sulem [27], pp. 229-230). Since ∇q_0 and $(\diamond) q_0(\diamond)$ lie in weak- L^2 but not L^2 , Ozawa's results suggest that $H^{1,1}(\mathbb{R}^2)$ is a natural limit for the inverse scattering method.

Our proof exploits the completely integrable method for the defocussing DS II equation developed by Fokas [20], Ablowitz-Fokas [2, 3, 4], Beals-Coifman [7, 8, 9], and Sung [28]. Explicitly, we exploit the *scattering transform* \mathcal{R} defined on $q \in \mathcal{S}(\mathbb{R}^2)$ by

$$(1.4) \quad (\mathcal{R}q)(k) = -\frac{1}{\pi} \int e_k(\zeta) q(\zeta) \overline{\mu_1(\zeta, k)} dm(\zeta).$$

Here $k \in \mathbb{C}$, $z = x + iy$, dm is Lebesgue measure on \mathbb{R}^2 , and e_k is the unimodular function

$$(1.5) \quad e_k(z) = e^{\bar{k}z - kz}.$$

The function μ_1 solves a $\bar{\partial}$ -problem in the z -variable with q as $\bar{\partial}$ -data, described in what follows (see (1.11)).

The map \mathcal{R} defines a continuous, and continuously invertible map from $\mathcal{S}(\mathbb{R}^2)$ onto itself (for a complete proof see Sung [28]). The inverse \mathcal{I} is given by

$$(1.6) \quad (\mathcal{I}r)(z) = -\frac{1}{\pi} \int e_{-k}(z) r(k) \nu_1(z, k) dm(k)$$

where ν_1 solves a $\bar{\partial}$ -problem in the k -variable with r as $\bar{\partial}$ -data (see (1.12)). The solutions μ_1 and ν_1 are normalized so that $\mu_1(z, k) \rightarrow 1$ as $|z| \rightarrow \infty$, and $\nu_1(z, k) \rightarrow 1$ as $|k| \rightarrow \infty$, and so that $\mu_1 = \nu_1 = 1$ when $q = r = 0$. Thus the linearizations of \mathcal{R} and \mathcal{I} at $q = 0$ are the maps

$$(1.7) \quad (\mathcal{F}\psi)(k) = -\frac{1}{\pi} \int e_k(\zeta) \psi(\zeta) dm(\zeta),$$

$$(1.8) \quad (\mathcal{F}^{-1}\psi)(z) = -\frac{1}{\pi} \int e_{-\zeta}(z) \psi(\zeta) dm(\zeta).$$

which differ from the usual Fourier maps by a sign and a linear transformation of coordinates.

The function

$$(1.9) \quad q(z, t) = \mathcal{I} \left(e^{4it \operatorname{Re}((\diamond)^2)} (\mathcal{R}q_0)(\diamond) \right) (z)$$

solves the Cauchy problem for (1.1) with initial data $q_0 \in \mathcal{S}(\mathbb{R}^2)$ (see Appendix B for a proof). Since

$$(r, t) \mapsto e^{4it \operatorname{Re}((\diamond)^2)} r(\diamond)$$

is a locally Lipschitz continuous map from $H^{1,1}(\mathbb{R}^2) \times \mathbb{R}$ to $H^{1,1}(\mathbb{R}^2)$, the key technical problem is to prove that \mathcal{R} and \mathcal{I} extend to locally Lipschitz continuous maps from $H^{1,1}(\mathbb{R}^2)$ to itself. We will prove:

Theorem 1.4. *The maps \mathcal{R} and \mathcal{I} are locally Lipschitz continuous maps from $H^{1,1}(\mathbb{R}^2)$ to $H^{1,1}(\mathbb{R}^2)$. Moreover, $\mathcal{I} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{I} = I$, where I is the identity on $H^{1,1}(\mathbb{R}^2)$. Finally, $\|\mathcal{R}(q)\|_2 = \|q\|_2$ and $\|\mathcal{I}(r)\|_2 = \|r\|_2$.*

Theorem 1.1 is an immediate consequence of Theorem 1.4 and the fact that (1.9) solves the DS II equation for Schwartz class initial data.

To discuss the proof of Theorem 1.4, we recall the Dirac-type linear spectral problem at zero energy associated to the DS II equation. Consider the problem

$$(1.10) \quad \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

To define the scattering transform \mathcal{R} , we seek a family of solutions of (1.10) parameterized by $k \in \mathbb{C}$ and satisfying the asymptotic condition

$$\lim_{|z| \rightarrow \infty} (e^{-kz} \psi_1, e^{-\bar{k}\bar{z}} \psi_2) = (1, 0).$$

Letting

$$\mu_1 = e^{-kz} \psi_1, \quad \mu_2 = e^{-\bar{k}\bar{z}} \psi_2.$$

we see that (1.10) is equivalent to

$$(1.11) \quad \begin{aligned} \bar{\partial} \mu_1 &= \frac{1}{2} e_k q \overline{\mu_2}, \\ \bar{\partial} \mu_2 &= \frac{1}{2} e_k q \overline{\mu_1}, \\ \lim_{|z| \rightarrow \infty} (\mu_1, \mu_2) &= (1, 0) \end{aligned}$$

This equation has a unique solution in $L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ for each $k \in \mathbb{C}$ provided that $q \in L^2(\mathbb{R}^2)$ (see, for example, [13]). If, also, $q \in L^p(\mathbb{R}^2)$ for $p > 2$, the solutions are Hölder continuous. For $q \in \mathcal{S}(\mathbb{R}^2)$, we recover $r = \mathcal{R}q$ from

$$\mu_2(z, k) \sim \frac{1}{z} \left(\frac{1}{2} r(k) \right) + \mathcal{O}(|z|^{-2}).$$

The integral formula (1.4) follows by recalling that, if u is a weak solution of $\bar{\partial}u = f$ that vanishes at infinity and f is rapidly decreasing,

$$u(z) = -\frac{1}{\pi z} \int f(\zeta) \, dm(\zeta) + \mathcal{O}(|z|^{-2}),$$

where dm is Lebesgue measure on \mathbb{R}^2 . Thus the $\bar{\partial}$ -data q determines $\mu = (\mu_1, \mu_2)$, whose asymptotics in turn determine r . Thus, to analyze the map \mathcal{R} , we need good estimates and large- k expansions for the solutions of the $\bar{\partial}$ -problem (1.11)

On the other hand, as shown in §4, the functions $(\nu_1, \nu_2) = (\mu_1, e_k \overline{\mu_2})$ are determined by the $\overline{\partial}_k$ -data r since the system

$$(1.12) \quad \begin{aligned} \overline{\partial}_k \nu_1 &= \frac{1}{2} e_k \overline{r \nu_1}, \\ \overline{\partial}_k \nu_2 &= \frac{1}{2} e_k \overline{r \nu_2}, \\ \lim_{|k| \rightarrow \infty} (\nu_1, \nu_2) &= (1, 0), \end{aligned}$$

where $\overline{\partial}_k$ differentiates in the k variable, also has a unique solution in $L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ for each $z \in \mathbb{C}$ provided that $r \in L^2(\mathbb{R}^2)$. For $r \in \mathcal{S}(\mathbb{R}^2)$, the functions $\nu = (\nu_1, \nu_2)$ determine $q = \mathcal{I}r$ through the asymptotic relation

$$\overline{\nu_2(z, k)} \sim \frac{1}{\overline{k}} \left(\frac{1}{2} q(z) \right) + \mathcal{O}(|k|^{-2})$$

and the integral formula (1.6) follows as before. Thus, to analyze the map \mathcal{I} we need large- z expansions and good estimates for solutions of (1.12).

Using the solution formula (1.9), we can also obtain large-time asymptotics of the solution for $r_0 = \mathcal{R}q_0 \in H^{1,1}$. The crux of the issue is to obtain fine estimates on solutions to the time-dependent $\overline{\partial}$ -problem

$$(1.13) \quad \begin{aligned} \overline{\partial}_k \nu_1 &= \frac{1}{2} e^{-itS} \overline{r_0 \nu_2}, \\ \overline{\partial}_k \nu_2 &= \frac{1}{2} e^{-itS} \overline{r_0 \nu_1}, \\ \lim_{|k| \rightarrow \infty} (\nu_1, \nu_2) &= (1, 0) \end{aligned}$$

where

$$(1.14) \quad S(z, k, t) = \frac{kz - \overline{k}\overline{z}}{it} + 4 \operatorname{Re}(k^2)$$

is a real-valued phase function with a single nondegenerate critical point at $k_c = iz/4t$. The solution is recovered via

$$(1.15) \quad q(z, t) = -\frac{1}{\pi} \int_{\mathbb{R}^2} e^{itS(k)} r(k) \nu_1(z, k, t) \, dm(k).$$

By obtaining large-time asymptotics of ν_1 , we will prove:

Theorem 1.5. *Suppose that $r \in H^{1,1}(\mathbb{R}^2)$. The solution q of the DS II equations obeys the asymptotic formula*

$$q(z, t) = u(z, t) + o(t^{-1})$$

in L_z^∞ -norm, where

$$u(z, t) = \mathcal{F}^{-1} \left(e^{4it \operatorname{Re}(\diamond^2)} r(\diamond) \right)$$

Remark 1.6. Thus, as claimed, the leading asymptotics are determined by the linear problem (1.2) with initial data $u_0 = \mathcal{F}^{-1}(\mathcal{R}q_0)$. The solution of (1.2) is $O(t^{-1})$ by explicit Fourier analysis of the linear problem, so that the remainder is indeed of lower order.

The results of Theorem 1.5 were first obtained by Kiselev [22] (see also [23], Theorem 7), but with a “small data” restriction and more stringent integrability and regularity assumptions. Kiselev’s analysis relies in part on separate asymptotic expansions of the solution $\nu_1(z, k, t)$ in the ‘exterior region’ $|k - k_c| \geq t^{-1/4}$ and in the ‘interior region’ $|k - k_c| < 2t^{1/4}$ with matching in the transition region.

In our proof, we remove Kiselev’s small data restriction in the defocussing case and replace the asymptotic expansions with a finer analysis of the integral operator M (see (5.1)) used to solve (1.13). Our analysis rests on scaling arguments and on the simple integration by parts formula (2.7) previously used by Bukhgeim [16] in his analysis of the inverse conductivity problem.

Inverse scattering for the Davey-Stewartson II equation was studied by Fokas [20], Ablowitz-Fokas [2, 3, 4], Beals-Coifman [7, 8, 9], Sung [28], and Brown [13]. Beals and Coifman construct global solutions for the defocussing DS II equation with initial data in $\mathcal{S}(\mathbb{R}^2)$ by inverse scattering methods, while Sung constructs solutions for the same case if $q \in L^p$ for some $p \in [1, 2)$ and the Fourier transform of q lies in $L^1 \cap L^\infty$ (see paper III of [28]). Sung [29] obtained the leading t^{-1} decay rate (but not the asymptotic formula) for solutions of the DS II equation with Schwarz class initial data, using his earlier work [28] on the inverse scattering method for DS II. Sung and Brown construct solutions to the focussing DS II equations with small initial data in L^2 ; the small data hypothesis avoids soliton solutions and the blow-up phenomena discussed in Remark 1.3. Brown actually shows Lipschitz continuity of the solution operator for small Cauchy data in L^2 for either the focussing or defocussing DS II equation. For the reasons stated in Remark 1.3, we believe that the space $H^{1,1}(\mathbb{R}^2)$ is an especially natural setting for global well-posedness results without small data restrictions.

The contents of this paper are as follows. In §2, we recall some basic facts related to $\bar{\partial}$ -problems and associated integral operators. In §3, we obtain asymptotic expansions for solutions of (1.11) and (1.12) which we apply in §4 to study the direct and inverse scattering maps, proving Theorem 1.4. We prove Theorem 1.5 in §5. Appendix A, written by Michael Christ, proves Brown’s multilinear estimate (see Proposition 2.3) by the methods of Bennett, Carbery, Christ, and Tao [10, 11]. In Appendix B we present, for the reader’s convenience, a concise proof that the inverse scattering formula (1.9) gives a classical solution of the DS II equation for initial data in $\mathcal{S}(\mathbb{R}^2)$. Appendix C computes large- z asymptotic expansions for solutions of (1.12) that are used in Appendix B.

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2. PRELIMINARIES

Notation. We will denote by $\|\cdot\|_p$ the usual L^p -norm, by p' the conjugate exponent $p/(p-1)$, and by $\|\cdot\|_{\mathcal{B}(X,Y)}$ the norm on bounded operators from the

Banach space X to the Banach space Y . We write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$. The notation $f(\cdot, \cdot)$ indicates a function of z and k with generic arguments.

We'll write L_z^p or L_k^p for L^p -spaces of functions with respect to the z or k variable. $C_0(\mathbb{R}^2)$ is the space of continuous complex-valued functions that vanish at infinity. We will often use the fact that $H^{1,1}(\mathbb{R}^2)$ is continuously embedded in $L^p(\mathbb{R}^2)$ for any $p \in (1, \infty)$. Finally, we denote by $L^{2,\theta}(\mathbb{R}^2)$ the space of measurable functions f on \mathbb{R}^2 with $\left\| (1 + |\cdot|)^\theta f(\cdot) \right\|_2 < \infty$.

We denote by $\langle \cdot, \cdot \rangle$ the pairing

$$(2.1) \quad \langle f, g \rangle = -\frac{1}{\pi} \int_{\mathbb{R}^2} \overline{f(z)} g(z) \, dm(z).$$

Adjoint A^* of linear operators A are taken with respect to this pairing.

Cauchy Transforms. The integral operators

$$P\psi = \frac{1}{\pi} \int \frac{1}{\zeta - z} f(\zeta) \, dm(\zeta),$$

$$\overline{P}\psi = \frac{1}{\pi} \int \frac{1}{\overline{\zeta} - \overline{z}} f(\zeta) \, dm(\zeta)$$

are formal inverses respectively of $\overline{\partial}$ and ∂ . We denote by P_k and \overline{P}_k the corresponding formal inverses of $\overline{\partial}_k$ and ∂_k . The following estimates are standard (see Vekua [30] or [6], §4.3).

Lemma 2.1. *Suppose that $2 < p < \infty$.*

- (i) *For any $f \in L^p$, $\|Pf\|_p \leq C_p \|f\|_{2p/(p+2)}$.*
- (ii) *For any q with $1 < q < 2$ and any $f \in L^p \cap L^q$, $\|Pf\|_\infty \leq C_{p,q} \|f\|_{L^p \cap L^q}$ and Pf is Hölder continuous of order $(p-2)/p$ with*

$$|(Pf)(z) - (Pf)(w)| \leq C_p |z - w|^{(p-2)/p} \|f\|_p.$$

- (iii) *For $2 < p < q$ and $u \in L^s$ for $q^{-1} + 1/2 = p^{-1} + s^{-1}$,*

$$\|P(u\psi)\|_q \leq C_{p,q} \|u\|_s \|\psi\|_p.$$

Remark 2.2. If $p > 2$ and $u \in L^s$ for $s \in (1, \infty)$, then estimate (iii) holds true for any $q > 2$.

If u is a continuous function vanishing at infinity, if u is a weak solution of $\overline{\partial}u = f$, and $\int |\zeta|^2 f(\zeta) \, dm(\zeta)$ is finite, then $u = Pf$ and

$$(2.2) \quad u(z) = -\frac{1}{\pi z} \int f(\zeta) \, dm(\zeta) + \mathcal{O}(|z|^{-2}).$$

Ahlfors-Beurling Transform. The Ahlfors-Beurling transform,

$$(2.3) \quad (\mathcal{S}f)(z) = -\frac{1}{\pi} \int \frac{1}{(z-w)^2} f(w) \, dw$$

defined as a Calderon-Zygmund type singular integral, has the property that for $f \in C_0^\infty(\mathbb{R}^2)$ we have $\mathcal{S}(\overline{\partial}f) = \partial f$. The operator \mathcal{S} is a bounded operator on L^p for $p \in (1, \infty)$ (see for example [6], §4.5.2). This fact allows us to obtain L^p -estimates on ∂ -derivatives of functions of interest from L^p -estimates on $\overline{\partial}$ -derivatives.

Brascamp-Lieb type estimates. The following multilinear estimate, due to Russell Brown ([13], Lemma 3; see also Nie-Brown [14]), plays a crucial role in the analysis of solutions to (1.11) and (1.12). See Appendix A for a proof of the estimate by the methods of Bennett, Carbery, Christ and Tao [10, 11]. Define

$$\Lambda_n(\rho, q_0, q_1, \dots, q_{2n}) = \int_{\mathbb{C}^{2n+1}} \frac{|\rho(\zeta)| |q_0(z_0)| \dots |q_{2n}(z_{2n})|}{\prod_{j=1}^{2n} |z_{j-1} - z_j|} dm(z)$$

where $dm(z)$ is product measure on \mathbb{C}^{2n+1} and

$$(2.4) \quad \zeta = \sum_{j=0}^{2n} (-1)^j z_j.$$

Proposition 2.3. [13] *The estimate*

$$(2.5) \quad |\Lambda_n(\rho, q_0, q_1, \dots, q_{2n})| \leq C_n \|\rho\|_2 \prod_{j=0}^{2n} \|q_j\|_2$$

holds.

Remark 2.4. Let $T^{(j)}\psi = Pe_k q_j \bar{\psi}$ where $q_j \in L^2(\mathbb{R}^2)$. Consider the form

$$(2.6) \quad \left\langle 1, e_k q_0 T^{(1)} \dots T^{(2n)} 1 \right\rangle$$

which defines a function of k . Integrating (2.6) against a test function $\hat{\rho}$ in the k -variable and applying (2.5) shows that (2.6) defines an L^2 function of k with

$$\left\| \left\langle 1, e_k q_0 T^{(1)} \dots T^{(2n)} 1 \right\rangle \right\|_2 \leq C_n \prod_{j=0}^{2n} \|q_j\|_2,$$

Integration by parts. If φ is a smooth, real-valued phase function with isolated critical points, and f vanishes in a neighborhood of the critical points of φ , we have the identity (see Bukhgeim [16])

$$(2.7) \quad -\frac{1}{\pi} \int \frac{e^{i\varphi(\zeta)}}{z - \zeta} f(\zeta) dm(\zeta) = \frac{e^{i\varphi(z)}}{i\varphi_{\bar{z}}(z)} f(z) \\ + \frac{1}{\pi} \int \frac{e^{i\varphi(\zeta)}}{z - \zeta} \bar{\partial}_\zeta \left(\frac{1}{\varphi_{\bar{\zeta}}(\zeta)} f(\zeta) \right) dm(\zeta).$$

In particular,

$$(2.8) \quad (Pe_k f)(z) = \frac{e_k(z)}{\bar{k}} f(z) - \frac{1}{\pi \bar{k}} \int \frac{e_k(\zeta)}{z - \zeta} (\bar{\partial}_\zeta f)(\zeta) dm(\zeta).$$

3. AN OSCILLATORY $\bar{\partial}$ -PROBLEM

In this section we study solutions of the $\bar{\partial}$ -problem (1.11). The $\bar{\partial}_k$ -problem for (ν_1, ν_2) has the same structure so we do not discuss it explicitly. To solve (1.11), introduce the antilinear operator

$$(3.1) \quad T\psi = \frac{1}{2} P(e_k q \bar{\psi})$$

and note that

$$(3.2) \quad T^2\psi = \frac{1}{4}P(e_k q \overline{P}e_{-k} \overline{q}\psi)$$

is \mathbb{C} -linear. The following is an easy consequence of Lemma 2.1.

Lemma 3.1. *For any $q \in H^{1,1}(\mathbb{R}^2)$, any $k \in \mathbb{C}$, and any p, r with $2 < p \leq r \leq \infty$, $T : L^p \rightarrow L^r$ is a compact antilinear operator with $\|T\|_{\mathcal{B}(L^p, L^r)} \leq C_{p,r} \|q\|_{H^{1,1}}$.*

For the boundedness one uses Lemma 2.1 and the fact that if $q \in H^{1,1}$, then $q \in L^s$ for all $s \in (1, \infty)$. Compactness follows from uniform Hölder continuity and the uniform decay of $T\psi$ at spatial infinity. The next lemma is also standard but we give a sketch of the proof.

Lemma 3.2. *For any $q \in H^{1,1}$ and any $p \in (2, \infty]$, $\ker_{L^p}(I - T^2)$ is trivial.*

Proof. First, note that if $\psi \in \ker_{L^\infty}(I - T^2)$, it follows from the mapping properties of T^2 that, in fact, $\psi \in L^p$ for any $p > 2$. So it suffices to consider $p \in (2, \infty)$. First, we note that $w \in \ker_{L^p}(I - T^2)$ if and only if there is a distribution solution $(u_1, u_2) = (w, Tw) \in L^p \times L^p$ for the system

$$\begin{aligned} \overline{\partial}u_1 &= \frac{1}{2}e_k q \overline{u_2}, \\ \overline{\partial}u_2 &= \frac{1}{2}e_k q \overline{u_1}. \end{aligned}$$

This in turn is true if and only if $u_\pm = (u_1 \pm u_2)$ satisfy $\overline{\partial}u_\pm = \pm \frac{1}{2}e_k q \overline{u_\pm}$. Note that, firstly, $q \in L^2$, and second, that $u_\pm \in L^p$ for $p > 2$, and hence $u_\pm \in L^2_{\text{loc}}$. It now follows from standard vanishing theorems for the $\overline{\partial}$ -problem (for the version used here, see Brown-Uhlmann [15], Corollary 3.11) that $u_\pm = 0$, hence $w = 0$. \square

We will also use the following estimate on the large- k behavior of $\|T^2\|_{\mathcal{B}(L^p)}$.

Lemma 3.3. *For any $p \in (2, \infty)$,*

$$\|T^2\|_{\mathcal{B}(L^p)} \leq C_p (1 + |k|)^{-1} \|q\|_{H^{1,1}}^2.$$

Proof. For the integral operator $T\psi = P e_k q \overline{\psi}$ we recover from (2.8) the useful estimate

$$(3.3) \quad \|T\varphi\|_p \leq C_p |k|^{-1} \left(\|q\varphi\|_p + \|\overline{\partial}_\zeta(q\varphi)\|_{\frac{2p}{p+2}} \right).$$

From (3.3) we conclude that

$$(3.4) \quad \|T(T\psi)\|_p \leq C_p |k|^{-1} \left(\|q \overline{T\psi}\|_p + \|(\overline{\partial}_\zeta q) \overline{T\psi}\|_{\frac{2p}{p+2}} + \|q \overline{\partial}_\zeta(\overline{T\psi})\|_{\frac{2p}{p+2}} \right)$$

To bound the first right-hand term in (3.4), we estimate, for s large, $r > p > 2$, and $1/r + 1/s = 1/p$

$$\|q \overline{T\psi}\|_p \leq \|q\|_s \|T\|_{\mathcal{B}(L^p, L^r)} \|\psi\|_p \leq C_p \|q\|_{H^{1,1}}^2 \|\psi\|_p.$$

The second right-hand term in (3.4) is bounded by $\|(\bar{\partial}_\zeta q)\|_2 \|q\|_2 \|\psi\|_p$. Finally, to estimate the third right-hand term in (3.4), we bound (recall \mathcal{S} is the Ahlfors-Beurling operator)

$$\|q \bar{\partial}_\zeta (\overline{T\psi})\|_{\frac{2p}{p+2}} \leq \|q\|_p \|\mathcal{S}(e_k q \psi)\|_2 \leq C_p \|q\|_{H^{1,1}}^2 \|\psi\|_p.$$

Collecting these estimates, we recover the desired bound. \square

From these facts, we can obtain uniform bounds and continuity of the resolvent $(I - T^2)^{-1}$.

Lemma 3.4. *The resolvent $(I - T^2)^{-1}$ is a bounded operator from L^p to itself for all $p \in (2, \infty)$, all $k \in \mathbb{C}$, and all $q \in H^{1,1}$. Moreover, $\sup_k \|(I - T)^{-1}\|_{\mathcal{B}(L^p)}$ is uniformly bounded for q in a fixed bounded subset of $H^{1,1}(\mathbb{R}^2)$. Finally, the map*

$$\begin{aligned} \mathbb{C} \times H^{1,1} &\rightarrow \mathcal{B}(L^p) \\ (k, q) &\mapsto (I - T^2)^{-1} \end{aligned}$$

is locally Lipschitz continuous in (k, q) with Lipschitz constant uniform in $k \in \mathbb{C}$ and q in a bounded subset of $H^{1,1}(\mathbb{R}^2)$.

Proof. We will (temporarily) use the notation $T_{k,q}$ for T to emphasize its dependence on k, q . For any $\theta \in [0, 1]$, the estimate

$$(3.5) \quad \|T_{k,q} - T_{k',q'}\|_{\mathcal{B}(L^p)} \leq C_p \left(\|(1 + |\cdot|)^\theta q(\cdot)\|_2 |k - k'|^\theta + \|q - q'\|_2 \right)$$

holds, showing that $(k, q) \mapsto T_{k,q}$ is a continuous map from $\mathbb{C} \times L^{2,\theta}$ to itself. Existence of $(I - T_{k,q}^2)^{-1}$ for given (k, q) follows from the Fredholm alternative, Lemma 3.1, and Lemma 3.2. From the second resolvent formula and the continuity of $(k, q) \mapsto T_{k,q}$, we conclude that the map $(k, q) \mapsto (I - T_{k,q}^2)^{-1}$ is also continuous from $\mathbb{C} \times L^{2,\theta}$ to $\mathcal{B}(L^p)$ for any $\theta \in [0, 1)$ and $p \in (2, \infty)$.

For q in a fixed bounded subset of $H^{1,1}$, $\|(I - T_{k,q}^2)^{-1}\|_{\mathcal{B}(L^p)}$ is bounded uniformly for $|k| > C_0$ owing to Lemma 3.3. On the other hand, the set B of (k, q) with $|k| \leq C_0$ and q in a fixed bounded subset of $H^{1,1}$ is compact in $\mathbb{C} \times L^{2,\theta}$. It follows that $\sup_{(k,q) \in B} \|(I - T_{k,q}^2)^{-1}\|_{\mathcal{B}(L^p)}$ is finite. This shows that $\sup_k \|(I - T_{k,q}^2)^{-1}\|_{\mathcal{B}(L^p)}$ is bounded for q in bounded subsets of $H^{1,1}(\mathbb{R}^2)$.

With the uniform resolvent bound, we can strengthen the continuity result. Indeed, for q, q' in a fixed bounded subset of $H^{1,1}(\mathbb{R}^2)$,

$$\|(I - T_{k,q}^2)^{-1} - (I - T_{k',q'}^2)^{-1}\|_{\mathcal{B}(L^p)} \leq 2C_p M_1^2 \|q\|_{H^{1,1}} \|T_{k,q} - T_{k',q'}\|_{\mathcal{B}(L^p)}$$

where M_1 bounds the resolvent. Taking $\theta = 1$ in (3.5) we obtain

$$\begin{aligned} \left\| (I - T_{k,q}^2)^{-1} - (I - T_{k',q'}^2)^{-1} \right\|_{\mathcal{B}(L^p)} &\leq 2C_p M_1^2 (1 + \|q\|_{H^{1,1}})^2 \\ &\quad \times (|k - k'| + \|q - q'\|_{H^{1,1}}) \end{aligned}$$

which gives the Lipschitz mapping property. \square

Lemma 3.5. *For any $k \in \mathbb{C}$ and $q \in H^{1,1}(\mathbb{R}^2)$, the problem (1.11) has a unique solution $\mu \in L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ with $(\mu_1 - 1, \mu_2) \in L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)$ for all $p > 2$. The solution μ is Hölder continuous in z of order α for any $\alpha \in (0, 1)$, and, for any $p > 2$, the map*

$$\begin{aligned} H^{1,1}(\mathbb{R}^2) \times \mathbb{C} &\rightarrow [C_0(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)]^2 \\ (q, k) &\mapsto (\mu_1 - 1, \mu_2) \end{aligned}$$

is locally Lipschitz continuous. Finally, for any $z \in \mathbb{C}$,

$$(3.6) \quad \lim_{|k| \rightarrow \infty} \mu(z, k) = (1, 0)$$

Proof. Uniqueness is immediate from Lemma 3.4. From the formula

$$(3.7) \quad (\mu_1 - 1, \mu_2) = \left(T^2 (I - T^2)^{-1} 1, T (I - T^2)^{-1} 1 \right),$$

the mapping properties of $(I - T^2)^{-1}$, and the second resolvent formula, we easily obtain the claimed continuity in z and continuity in (q, k) . Note that we can use L^p continuity of the resolvent plus smoothing properties of the operators T^2 and T in (3.7) to obtain continuity as maps from $H^{1,1}(\mathbb{R}^2) \times \mathbb{C}$ to $C_0(\mathbb{R}^2)$. Finally, by the continuity of μ , it suffices to show that (3.6) holds for $q \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. For such q , we can use the integration by parts formula to compute

$$T1 = \bar{k}^{-1} (e_k q - T(\partial q))$$

and conclude that $(T1)(z, k) \rightarrow 0$ as $|k| \rightarrow \infty$, uniformly in z , so that (3.6) holds. \square

We will also need the following expansion formula for μ_1 . An analogous expansion for μ_2 follows since $\mu_2 = T\mu_1$.

Lemma 3.6. *For any positive integer N and any $p > 2$, the formula*

$$(3.8) \quad \mu_1 - 1 = \sum_{j=1}^N T^{2j} 1 + \mu_1^{(N)}$$

holds in $L^p(\mathbb{R}^2) \cap C_0(\mathbb{R}^2)$, where the map

$$\begin{aligned} H^{1,1}(\mathbb{R}^2) &\rightarrow L_k^\infty(L_z^p(\mathbb{R}^2)) \\ q &\mapsto (1 + |\diamond|)^N \mu_1^{(N)}(\cdot, \diamond) \end{aligned}$$

is locally Lipschitz continuous.

Proof. The expansion (3.8) holds in $L^p \cap C_0$ with

$$\mu_1^{(N)} = (I - T^2)^{-1} T^{2N} (T^2 1).$$

From Lemmas 3.3 and 3.4, we have the estimate

$$\left\| (1 + |\diamond|)^N \mu_1^{(N)}(\cdot, \diamond) \right\|_{L_k^\infty(L_z^p)} \leq C(q) \|T^2 1\|_p$$

so that $(1 + |\diamond|)^N \mu_1^{(N)}(\cdot, \diamond) \in L_k^\infty(L_z^p)$ as claimed. The Lipschitz continuity follows from the continuity of the maps $(k, q) \rightarrow T^2$, and $(k, q) \rightarrow (I - T^2)^{-1}$ as maps from $\mathbb{C} \times H^{1,1}$ to $\mathcal{B}(L^p)$ and the continuity of $(k, q) \rightarrow T^2 1$ as maps from $\mathbb{C} \times H^{1,1} \rightarrow L^p$. \square

4. DIRECT AND INVERSE SCATTERING TRANSFORMS

We now construct the map \mathcal{R} by working initially on $\mathcal{S}(\mathbb{R}^2)$ and extending by continuity to $H^{1,1}(\mathbb{R}^2)$. It is known that \mathcal{R} is an isomorphism from $\mathcal{S}(\mathbb{R}^2)$ to itself (see Sung [28] for a complete proof) but we do not use this fact.

Lemma 4.1. *The map \mathcal{R} with domain $\mathcal{S}(\mathbb{R}^2)$ extends to a locally Lipschitz continuous map from $H^{1,1}(\mathbb{R}^2)$ to $L^{2,1}(\mathbb{R}^2)$.*

Proof. It suffices to check the Lipschitz continuity for q in the dense subset $\mathcal{S}(\mathbb{R}^2)$ of $H^{1,1}(\mathbb{R}^2)$. First, we check continuity from $H^{1,1}(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$. Using the expansion (3.8) in the representation formula we have

$$r(k) = (\mathcal{F}q)(k) + \sum_{j=1}^N \langle e_{-k} \bar{q}, T^{2j} 1 \rangle + \langle e_{-k} \bar{q}, \mu_1^{(N)} \rangle$$

The first right-hand (Fourier) term is Lipschitz continuous, the second term is a continuous map by Remark 2.4, and the third is a locally Lipschitz continuous map into L^2 for $N \geq 2$ since $q \in L^{p'}$ for any $p > 2$ and Lemma 3.6 holds.

Next, we study the map $q \mapsto (\diamond)r(\diamond)$. Integrating by parts and using the equation $\bar{\partial}\mu_1 = e_k q \bar{\mu}_2$, we obtain

$$kr(k) = I_1 + I_2$$

where

$$I_1 = \frac{1}{\pi} \int e_k(\zeta) (\partial_\zeta q)(\zeta) \overline{\mu_1(\zeta, k)} dm(\zeta),$$

$$I_2 = \frac{1}{2\pi} \int |q(\zeta)|^2 \mu_2(\zeta, k) dm(\zeta).$$

To analyze I_2 , we use the expansion

$$(4.1) \quad \mu_2 = \sum_{j=1}^N T^{2j+1} 1 + (I - T^2)^{-1} T^{2N+3} 1.$$

The term in I_2 corresponding to the remainder term in (4.1) has rapid decay in $|k|$ and the required continuity. For the remaining terms we compute

$$\langle |q|^2, T^{2j+1} 1 \rangle = \langle |q|^2, P(e_k q(T^{2j} 1)) \rangle = \langle e_{-k} \bar{q} \bar{P}(|q|^2), T^{2j} 1 \rangle$$

which is controlled by Remark 2.4, where $q_0 = \overline{qP}(|q|^2) \in L^2$.

To analyze I_1 , let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\eta(z) = 1$ for $|z| \leq 1$ and $\eta(z) = 0$ for $|z| \geq 2$. Then

$$I_1 = I_{11} + I_{12}$$

where

$$\begin{aligned} I_{11} &= \frac{1}{\pi} \int e_k(\zeta) \eta(\zeta) (\partial_\zeta q)(\zeta) \overline{\mu_1(\zeta, k)} dm(\zeta), \\ I_{12} &= \frac{1}{\pi} \int e_k(\zeta) (\partial_\zeta q)(\zeta) (1 - \eta(\zeta)) \overline{\mu_1(\zeta, k)} dm(\zeta). \end{aligned}$$

Since $\eta \partial_\zeta q \in L^{p'}$ for any $p > 2$, we can show that I_{11} defines an L^2 function of k with the claimed continuity properties by mimicking the proof that $r \in L^2$. To analyze I_{12} , let $w = (1 - \eta) \partial q$, note that $\zeta^{-1} w(\zeta) \in L^{p'}$ for $p > 2$, and write

$$(4.2) \quad I_{12} = \mathcal{F}(w) + \frac{1}{\pi} \int e_k(\zeta) w(\zeta) \left(\overline{\mu_1(\zeta, k)} - 1 \right) dm(\zeta).$$

The first term has the correct mapping properties. To analyze the second term, expand

$$(4.3) \quad \begin{aligned} \mu_1(\zeta, k) - 1 &= \frac{1}{\pi \zeta} \int e_k(\zeta') q(\zeta') \mu_2(\zeta', k) dm(\zeta') \\ &\quad + \frac{1}{\pi \zeta} \int \frac{e_k(\zeta')}{\zeta - \zeta'} \zeta' q(\zeta') \mu_2(\zeta', k) dm(\zeta'). \end{aligned}$$

Inserting the second right-hand term in (4.3) into (4.2) leads to an integral that can be analyzed along the same lines as I_{11} . Inserting the first right-hand term in (4.3) into (4.2) gives

$$\frac{1}{\pi^2} \int e_k(\zeta) (\partial_\zeta q)(\zeta) \zeta^{-1} (1 - \eta(\zeta)) dm(\zeta) \times \int e_k(\zeta') q(\zeta') \mu_2(\zeta', k) dm(\zeta').$$

Since the first factor is the Fourier transform of an L^2 function, it suffices to show that the second factor is an L^∞ function of k . This follows from the facts that $q \in L^{p'}$ and $\mu_2(\cdot, k) \in L^p$ uniformly in k . \square

Now we show that, if μ solves (1.11) and $\nu = (\mu_1, e_k \overline{\mu_2})$, then ν satisfies (1.12).

Lemma 4.2. *Suppose that $q \in H^{1,1}(\mathbb{R}^2)$. and let $\nu = (\mu_1, e_k \overline{\mu_2})$. Then, ν is differentiable for a.e. k and*

$$(4.4) \quad \begin{aligned} \overline{\partial}_k \nu_1 &= \frac{1}{2} e_k \overline{r(k)} \overline{\nu_2}, \\ \overline{\partial}_k \nu_2 &= \frac{1}{2} e_k \overline{r(k)} \overline{\nu_1}, \\ \lim_{|k| \rightarrow \infty} \nu(z, k) &= (1, 0). \end{aligned}$$

Proof. First, suppose that $q \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. It follows from (1.4) and the continuity properties of μ_1 that $r = \mathcal{R}q$ is a bounded continuous function of k .

For a function $f(k, \bar{k})$, let

$$\begin{aligned} (\Delta_h f)(k, \bar{k}) &= \frac{1}{h} [f(k+h, \bar{k}) - f(k, \bar{k})], \\ (\overline{\Delta_h f})(k, \bar{k}) &= \frac{1}{\bar{h}} [f(k, \bar{k}+\bar{h}) - f(k, \bar{k})], \end{aligned}$$

and let

$$\begin{aligned} \rho_1 &= (\Delta_h + z)\mu_2, \\ \rho_2 &= \overline{\Delta_h}\mu_1. \end{aligned}$$

It follows from the equations $\mu_1 = 1 + T\mu_2$, $\mu_2 = T\mu_1$ that

$$\begin{aligned} \rho_1 &= T\rho_2 + \frac{1}{2}r(k) + \Phi_1 \\ \rho_2 &= \Phi_2 + T\rho_1, \end{aligned}$$

where $\Phi_1, \Phi_2 \rightarrow 0$ in $L_z^p(\mathbb{R}^2) \cap C_0(\mathbb{R}^2)$ as $h \rightarrow 0$, uniformly in k . For $q \in C_0^\infty(\mathbb{R}^2)$ we also have

$$z\mu_2(z, k) = \frac{1}{2}r(k) + \mathcal{O}(|z|^{-1})$$

so that $(\rho_1, \rho_2) \rightarrow \frac{1}{2}r(k)(1, 0)$ as $|z| \rightarrow \infty$. Since

$$\begin{aligned} \rho_1 &= \frac{1}{2}r(k) + T^2(\rho_1) + T\Phi_2 + \Phi_1, \\ \rho_2 &= T\rho_1 + \Phi_2, \end{aligned}$$

we can use $\mu = ((I - T^2)^{-1}1, T(I - T^2)^{-1}1)$ and the asymptotic condition on (ρ_1, ρ_2) to obtain the unique solution

$$\begin{aligned} \rho_1 &= \frac{1}{2}r(k)\mu_1 + \Psi_1, \\ \rho_2 &= \frac{1}{2}\overline{r(k)}\mu_2 + \Psi_2 \end{aligned}$$

for any $h \neq 0$, where $\Psi_1, \Psi_2 \rightarrow 0$ in $L_z^p(\mathbb{R}^2) \cap C_0(\mathbb{R}^2)$ as $h \rightarrow 0$ uniformly in k . We can now take limits as $h \rightarrow 0$ in $C(\mathbb{R}^2)$ to conclude that

$$\begin{aligned} (\partial_k + z)\mu_2 &= \frac{1}{2}r(k)\mu_1, \\ \overline{\partial_k}\mu_1 &= \frac{1}{2}\overline{r(k)}\mu_2. \end{aligned}$$

Setting $\nu = (\mu_1, e_k\overline{\mu_2})$ we see that (4.4) holds for all k .

Now suppose that $q \in H^{1,1}$ and $\{q_n\}$ is a sequence from $C_0^\infty(\mathbb{R}^2)$ with $q_n \rightarrow q$ in $H^{1,1}$. If $r_n = \mathcal{R}(q_n)$, we have $r_n \rightarrow r$ in $L^{2,1}$, and, by continuity of μ , we also have $\nu_n \rightarrow \nu$ in $L^\infty \times L^\infty$, where ν solves the integral equation form of (1.12). It now follows that (1.12) holds in distribution sense if $q \in H^{1,1}$, μ solves (1.11), and $\nu = (\mu_1, e_k\overline{\mu_2})$. \square

We can now prove:

Lemma 4.3. *The map \mathcal{R} is locally Lipschitz continuous from $H^{1,1}$ to itself.*

Proof. Given Lemma 4.1, it suffices show that the map $q \mapsto \partial r$ is locally Lipschitz continuous. For $q \in H^{1,1}$ we compute, for a.e. k ,

$$(\partial r)(k) = I_1 + I_2$$

where

$$I_1 = \frac{1}{\pi} \int e_k(\zeta) \zeta q(\zeta) \overline{\mu_1(\zeta, k)} dm(\zeta),$$

$$I_2 = -\frac{r(\zeta)}{2\pi} \int q(\zeta) \mu_2(\zeta, k) dm(\zeta)$$

and we have used the fact that $\overline{\partial}_k \mu_1(z, k) = \frac{1}{2} e_k(z) r(k) \overline{\mu_2(z, k)}$ for a.e. k . Since $r \in L^2$, $q \in L_z^{p'}$ and $\mu_2 \in L_z^p$ uniformly in k , it is clear that $q \mapsto I_2$ defines a locally Lipschitz continuous map from $H^{1,1}$ into L^2 . To analyze the first term, we use the same argument used to estimate I_1 in the proof of Lemma 4.1. \square

By a similar analysis, we can show that the map \mathcal{I} defined by (1.12) and (1.6), initially defined on $\mathcal{S}(\mathbb{R}^2)$, extends to a locally Lipschitz continuous map from $H^{1,1}$ to itself. Moreover, given $r \in H^{1,1}$ and the unique solution ν of (1.12), we can show that $\mu = (\nu_1, e_k \overline{\nu_2})$ satisfies (1.11) with q given by (1.6). Using the uniqueness of solutions to (1.11) and (1.12), we can prove:

Lemma 4.4. *The maps \mathcal{R} and \mathcal{I} are one-to-one and onto, and the equality $\mathcal{I} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{I} = I$ holds, where I is the identity map on $H^{1,1}(\mathbb{R}^2)$.*

Proof. Suppose that $\mathcal{R}q_1 = \mathcal{R}q_2$. Let $\mu^{(1)}, \mu^{(2)}$ be the respective (vector-valued) solutions to (1.11), and let $\nu^{(1)}, \nu^{(2)}$ be the corresponding solutions to (1.12). Clearly $\nu^{(1)} = \nu^{(2)}$ since $\mathcal{R}q_1 = \mathcal{R}q_2$, so $q_1 = q_2$ by the reconstruction formula (1.6). Given $r \in H^{1,1}$, we solve (1.12) for ν and construct $q \in H^{1,1}$ with $r = \mathcal{R}q$ using (1.6). Hence \mathcal{R} is onto and $\mathcal{I} \circ \mathcal{R} = I$. A similar proof shows that \mathcal{I} is one-to-one and onto with $\mathcal{R} \circ \mathcal{I} = I$. \square

Finally, we prove a Plancherel-type identity.

Lemma 4.5. *For q and r belonging to $\mathcal{S}(\mathbb{R}^2)$, the identities*

$$\|\mathcal{R}q\|_2 = \|q\|_2$$

and

$$\|\mathcal{I}r\|_2 = \|r\|_2$$

hold.

Proof. We'll prove the first identity since the second proof is similar. From the representation formula (1.4) we have

$$\begin{aligned} \int |r(k)|^2 dm(k) &= -\frac{1}{\pi} \int \int \overline{r(k)} e_k(\zeta) q(\zeta) \overline{\mu_1(\zeta, k)} dm(\zeta) dm(k) \\ &= \int q(\zeta) \overline{\left(-\frac{1}{\pi} \int e_{-k}(\zeta) r(k) \mu_1(\zeta, k) dm(k) \right)} dm(\zeta) \\ &= \int |q(\zeta)|^2 dm(\zeta) \end{aligned}$$

where in the last step we used (1.6). \square

Proof of Theorem 1.1. An immediate consequence of Lemmas 4.1-4.5. \square

5. LARGE-TIME ASYMPTOTICS

In this section we prove Theorem 1.5 by studying the $\bar{\partial}$ -problem (1.13). Let

$$(5.1) \quad M\psi = P_k e^{-itS} \bar{r} \bar{\psi}$$

where S is the phase function (1.14). Note that $\|M\|_{\mathcal{B}(L^p, L^q)} \leq C_{p,q} \|r\|_{H^{1,1}}$ for any $p, q > 2$, where $C_{p,q}$ is independent of (z, t) , and that $(I - M^2)^{-1}$ exists for any (z, t) . Thus,

$$\nu_1 = (I - M^2)^{-1} 1.$$

We will obtain good estimates for $\nu_1 - 1$ in L^p for $p > 2$ by showing that $\|M^2\|_{\mathcal{B}(L^p)}$ has an explicit rate of decay as $t \rightarrow \infty$ and obtaining estimates on $M^2 1$ up to an explicit term. This will suffice to obtain the leading asymptotics of $q = q(z, t)$ given by

$$(5.2) \quad q(z, t) = -\frac{1}{\pi} \int e^{itS(z, k, t)} r(k) \nu_1(z, k, t) dm(k)$$

in L_z^∞ , where $r = \mathcal{R}q_0$.

The phase function S has a single, nondegenerate critical point:

$$(5.3) \quad S(k, t, z) = S_0 + 4 \operatorname{Re} \left((k - k_c)^2 \right)$$

where

$$(5.4) \quad k_c = \frac{iz}{4t},$$

while

$$S_0 = -\frac{1}{4} \operatorname{Re} (z^2/t^2).$$

We will sometimes write $S(k)$ or simply S for $S(k, t, z)$. Note that

$$(5.5) \quad S_{\bar{k}} = 4 (\bar{k} - \bar{k}_c).$$

We define functions which localize near or away from the critical point k_c , with the former localization in a ball of radius $\mathcal{O}(t^{-1/4})$, as follows. Let $\eta \in C_0^\infty(\mathbb{R}^2)$ with $\eta(\xi) = 1$ for $|\xi| \leq 1$, and $\eta(\xi) = 0$ for $|\xi| \geq 2$. Fix (z, t) and let $\chi(k) = \eta(t^{1/4}(k - k_c))$ where k_c is given by (5.4). Thus χ has support near the critical point of the phase function, while $(1 - \chi)$ has support away from the critical point. The following estimates quantify the singularity of $S_{\bar{k}}^{-1}$ near the critical point. For any $\sigma > 2$, we have

$$(5.6) \quad \left\| S_{\bar{k}}^{-1} (1 - \chi) \right\|_\sigma \leq C_\sigma t^{1/4-1/(2\sigma)},$$

while for any $\sigma > 1$ we have

$$(5.7) \quad \left\| S_{\bar{k}}^{-2} (1 - \chi) \right\|_\sigma \leq C_\sigma t^{1/2-1/(2\sigma)},$$

and

$$(5.8) \quad \left\| S_{\bar{k}}^{-1} (\bar{\partial}_k \chi) \right\|_\sigma \leq C_\sigma t^{1/2-1/(2\sigma)}.$$

We also have, for any $\sigma > 1$,

$$(5.9) \quad \|\chi\|_\sigma \leq C_\sigma t^{-1/(2\sigma)}.$$

Finally, if $\phi \in \mathcal{C}_0^\infty$ with $\phi(k) = 1$ for $|k - k_c| \leq 1$ and $\phi(k) = 0$ for $|k - k_c| \geq 2$, we have

$$(5.10) \quad \left\| S_k^{-1} (1 - \chi) \phi \right\|_\sigma \leq C_\sigma t^{1/4 - 1/(2\sigma)}$$

for any $\sigma > 1$.

First we prove a general estimate on the operator M^2 .

Lemma 5.1. *Suppose that $p \in (2, \infty)$, $\varepsilon > 0$, and $r \in H^{1,1}$. The estimate*

$$\|M^2\|_{\mathcal{B}(L^p)} \leq C_{\varepsilon,p} \|r\|_{H^{1,1}} t^{\varepsilon - 1/4}$$

holds.

Proof. First, we note the integration by parts formula

$$(5.11) \quad M\psi = -\frac{e^{-itS}}{itS_k} \bar{r}\bar{\psi} + \frac{1}{\pi it} \int \frac{e^{-itS(\zeta)}}{k - \zeta} \bar{\partial}_k \left(S_k^{-1} \bar{r}\bar{\psi} \right) (\zeta) \, dm(\zeta)$$

true for ψ vanishing near $k = k_c$. We split

$$M^2\psi = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= M(1 - \chi)M\psi, \\ I_2 &= M\chi M\psi, \end{aligned}$$

Note that

$$\|\chi r\|_\sigma \leq C_\sigma t^{-1/(2\sigma_1)} \|r\|_{\sigma_2}$$

where $\sigma^{-1} = \sigma_1^{-1} + \sigma_2^{-1}$. In what follows, we take $\|\psi\|_p = 1$.

I_1 : Using (5.11) we get

$$\begin{aligned} \|I_1\|_p &\leq t^{-1} \left\| S_k^{-1} (1 - \chi) \right\|_{\sigma_1} \|r\|_{\sigma_2} \|M\psi\|_{\sigma_3} \\ &\quad + t^{-1} \left\| \bar{\partial}_k \left((1 - \chi) S_k^{-1} r M\psi \right) \right\|_{2p/(p+2)} \end{aligned}$$

where $\sigma_1^{-1} + \sigma_2^{-1} + \sigma_3^{-1} = p^{-1}$. The first right-hand term is bounded by a constant times $t^{-3/4 - 1/(2\sigma_1)} \|r\|_{H^{1,1}}^2$ for any $\sigma_1 > p$. The second term is estimated up to a constant by t^{-1} times the quantity

$$\begin{aligned} &\left\| S_k^{-2} (1 - \chi) \right\|_{\sigma_1} \|r\|_2 \|M\psi\|_{\sigma_2} + \left\| S_k^{-1} \bar{\partial}_k \chi \right\|_{\sigma_1} \|r\|_2 \|M\psi\|_{\sigma_2} \\ &+ \left\| S_k^{-1} (1 - \chi) \right\|_{\sigma_1} \left\| \bar{\partial} r \right\|_2 \|\psi\|_{\sigma_2} \end{aligned}$$

where $\sigma_1^{-1} + \sigma_2^{-1} = 1/p$. Each of the resulting terms is bounded by $t^{1/2 - 1/(2\sigma_1)} \|r\|_{H^{1,1}}^2$ so that

$$(5.12) \quad \|I_1\|_p \leq C_p t^{-1/2 - 1/\sigma}$$

for any $\sigma > p$.

I_2 : Estimate

$$(5.13) \quad \|I_2\|_p \leq C_p \|\chi r\|_{\sigma_1} \|r\|_{\sigma_2} \|\psi\|_p$$

with $\sigma_1^{-1} + \sigma_2^{-1} = 1/2$. Choosing $\sigma_1 > 2$ so that $1/(2\sigma_1) = \varepsilon - 1/4$ we obtain $\|I_2\| \leq C_{p,\varepsilon} t^{\varepsilon-1/4}$.

Combining (5.12) and (5.13) gives the required estimate. \square

As an immediate consequence, we have the following estimate.

Lemma 5.2. *Suppose that $r \in H^{1,1}$. For sufficiently large $t > 0$, any $\varepsilon > 0$, and any $p > 2$, the estimate*

$$\left\| (I - M^2)^{-1} M^{2j} \right\|_{\mathcal{B}(L^p)} \leq C(\varepsilon, p, r) t^{2j(\varepsilon-1/4)}$$

holds.

We obtain the needed estimates on $M^2 1$ in two steps.

Lemma 5.3. *Suppose $r \in H^{1,1}$ and $p > 2$. The estimate*

$$\|M1\|_p \leq C \|r\|_{H^{1,1}} t^{-3/4}$$

holds for all sufficiently large t .

Proof. We'll fix (z, t) and $k_c = iz/(4t)$, and obtain estimates uniform in (z, t) . First,

$$M1 = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= -\frac{1}{\pi} \int \frac{e^{-itS(\zeta)}}{k - \zeta} \overline{r(\zeta)} \chi(\zeta) \, dm(\zeta), \\ I_2 &= -\frac{1}{\pi} \int \frac{e^{-itS(\zeta)}}{k - \zeta} \overline{r(\zeta)} (1 - \chi(\zeta)) \, dm(\zeta). \end{aligned}$$

To estimate I_1 , we set $J_1(\zeta) = I_1(k_c + \zeta/\sqrt{t})$ and change variables to $\zeta = k_c + \xi/\sqrt{t}$ in I_1 to obtain

$$J_1(\zeta) = -\frac{e^{-itS_0}}{\pi\sqrt{t}} \int \frac{e^{-2i\operatorname{Re}(\xi^2)}}{\zeta - \xi} \eta\left(t^{-1/4}\xi\right) \overline{r\left(k_c + \xi/\sqrt{t}\right)} \, dm(\xi).$$

By Lemma 2.1(i),

$$\begin{aligned} \|J_1\|_p &\leq C_p t^{-1/2} \left\| \eta\left(t^{-1/4} \cdot\right) \right\|_\infty \left\| r\left(k_c + \cdot/\sqrt{t}\right) \right\|_{2p/(p+2)} \\ &\leq C_p t^{-1-1/p} \|r\|_p \end{aligned}$$

Using $\|I_1(\cdot)\|_p = t^{-1/p} \|J_1(\cdot)\|_p$ we conclude that

$$\|I_1\|_p \leq C_p t^{-1-2/p} \|r\|_{H^{1,1}}.$$

To estimate I_2 we use (5.11) to obtain

$$I_2 = I_{21} + I_{22} + I_{23} + I_{24}$$

where

$$\begin{aligned}
I_{21} &= -\frac{e^{-itS}}{itS_k} \bar{r} (1 - \chi), \\
I_{22} &= \frac{1}{4\pi it} \int \frac{e^{-itS(\xi)}}{k - \zeta} \frac{1}{(\bar{\zeta} - \bar{k}_c)^2} (1 - \chi(\zeta)) \overline{r(\zeta)} \, dm(\zeta), \\
I_{23} &= \frac{1}{4\pi it} \int \frac{e^{-itS(\xi)}}{k - \zeta} \frac{1}{\bar{\zeta} - \bar{k}_c} (-\bar{\partial}_\zeta \chi)(\zeta) \overline{r(\zeta)} \, dm(\zeta), \\
I_{24} &= -\frac{1}{4\pi it} \int \frac{e^{-itS(\xi)}}{k - \zeta} \frac{1}{\bar{\zeta} - \bar{k}_c} (1 - \chi(\zeta)) \overline{(\partial_\zeta r)}(\zeta) \, dm(\zeta).
\end{aligned}$$

Using (5.6) we estimate, for $\sigma^{-1} + \tau^{-1} = p^{-1}$ and $\sigma > 2$ (note $\tau > p$)

$$\begin{aligned}
\|I_{21}\|_p &\leq C_p t^{-1} \|r\|_\tau \left\| S_k^{-1} (1 - \chi(\cdot)) \right\|_\sigma \\
&\leq C_{p,\tau} t^{-3/4+1/(2\tau)-1/2p} \|r\|_{H^{1,1}}.
\end{aligned}$$

Next, introducing $\phi \in \mathcal{C}_0^\infty$ as in the discussion leading to (5.10), we have

$$(5.14) \quad \|I_{22}\|_p \leq C_p t^{-1} \left[\left\| S_k^{-2} \phi (1 - \chi(\cdot)) \right\|_{\sigma_1} \|r\|_{\tau_1} + \left\| S_k^{-2} (1 - \phi) \right\|_2 \|r\|_p \right].$$

In the first right-hand term of (5.14), we have $\sigma_1^{-1} + \tau_1^{-1} = 1/2 + p^{-1}$ and we may choose $1 < \sigma_1 < 2$ and $\tau_1 > p$ so that the first term is bounded by $C_p t^{-1/2-1/(2\sigma_1)} \|r\|_{\tau_1} \leq C_p t^{-3/4} \|r\|_{\tau_1}$. The second right-hand term is bounded uniformly in t . In a similar way, using (5.8),

$$\|I_{23}\|_p \leq C_p t^{-3/4+1/2(1/\tau-1/p)} \|r\|_\tau$$

and finally, using (5.6) again,

$$\|I_{24}\|_p \leq C t^{-3/4-1/2p} \|\bar{\partial} r\|_2.$$

Collecting these estimates we have

$$\|I_2\| \leq C t^{-3/4} \|r\|_{H^{1,1}}.$$

□

Next, we estimate $M^2 1$ up to an explicit term.

Lemma 5.4. *For any $r \in H^{1,1}$, $\varepsilon > 0$ sufficiently small, and $p > 2$, the relation*

$$\left\| M^2 1 + \frac{1}{4} t^{-1} P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r) \right\|_p \leq C_p \|r\|_{H^{1,1}}^2 t^{-1-\varepsilon}$$

holds, where \mathcal{S} is the Ahlfors-Beurling operator.

Proof. Write

$$M^2 1 = M(\chi \overline{M1}) + M((1 - \chi) \overline{M1})$$

First, for $\sigma^{-1} + \tau^{-1} = p^{-1}$

$$\begin{aligned} \|M\chi M1\|_p &\leq \|\chi r\|_\sigma \|M1\|_\tau \\ &\leq C_\tau t^{-3/4} \|r\|_{H^{1,1}} \|\chi\|_{\sigma_1} \|r\|_{\sigma_2} \\ &\leq C_\tau t^{-3/4-1/(2\sigma_1)} \|r\|_{H^{1,1}} \|r\|_{\sigma_2}. \end{aligned}$$

where $\sigma_1^{-1} + \sigma_2^{-1} + \tau^{-1} = 1/2 + 1/p$ and we've used $\|\chi\|_{\sigma_1} \leq Ct^{-1/(2\sigma_1)}$. We want to choose $\sigma_1 < 2$ so that $\|M\chi M1\|_p = O(t^{-1-\varepsilon})$ for any $p > 2$ and sufficiently small $\varepsilon > 0$. We also need $\tau > 2$ and $\sigma_2 > 1$. Let $\tau = p/(1-\delta)$. Then

$$\frac{1}{\sigma_1} + \frac{1}{\sigma_2} = \frac{1}{2} + \frac{\delta}{p}$$

and if $\sigma_1 = 2 - \varepsilon_1$ we have $\sigma_2^{-1} = \delta/p - \varepsilon/(4 - 2\varepsilon)$. Choose $\varepsilon \in (0, 4/(p+2))$ and $\delta \in (\varepsilon p/(4 - 2\varepsilon), 1)$.

Next, we'll estimate $\|M(1-\chi)M1\|_p$. Integrating by parts, we get

$$M(1-\chi)M1 = I_1 + I_2 + I_3 + I_4 + I_5$$

where

$$\begin{aligned} I_1 &= -\frac{1}{itS_k} e^{-itS} \bar{r} (1-\chi) \overline{M1}, \\ I_2 &= -\frac{1}{it} P_k \left(e^{-itS} \bar{\partial}_k \left(S_k^{-1} \right) (1-\chi) \bar{r} (\overline{M1}) \right), \\ I_3 &= \frac{1}{it} P_k \left(e^{-itS} S_k^{-1} (\bar{\partial}_k \chi) \bar{r} (\overline{M1}) \right), \\ I_4 &= -\frac{1}{it} P_k \left(e^{-itS} S_k^{-1} (1-\chi) (\bar{\partial}_k r) (\overline{M1}) \right) \\ I_5 &= \frac{1}{it} P_k \left(e^{-itS} S_k^{-1} \bar{r} (1-\chi) \mathcal{S}(e^{itS} r) \right). \end{aligned}$$

Notice that I_5 is the 'extra' term in the claimed estimate, so it suffices to bound I_1-I_4 .

In what follows, we'll introduce a cutoff function $\phi \in C_0^\infty$ with $\phi(k) = 1$ for $|k - k_c| \leq 1$ and $\phi(k) = 0$ for $|k - k_c| \geq 2$.

I_1 : Estimate

$$\begin{aligned} (5.15) \quad \|I_1\|_p &\leq C_p t^{-1} \left\| S_k^{-1} (1-\chi) \right\|_{\sigma_1} \|r\|_{\sigma_2} \|M1\|_{\sigma_3} \\ &\leq C_{p,\sigma} t^{-3/2-1/(2\sigma_1)} \|r\|_{H^{1,1}}^2 \end{aligned}$$

where $\sigma_1^{-1} + \sigma_2^{-1} + \sigma_3^{-1} = p^{-1}$, hence $\sigma_1 > p$.

I_2 : Estimate

$$\|I_2\|_p \leq C_p t^{-1} \left[\left\| S_k^{-2} (1-\chi) \phi \right\|_{\sigma_1} \|r\|_2 \|M1\|_{\sigma_2} + \|r\|_2 \|M1\|_p \right]$$

where $\sigma_1^{-1} + \sigma_2^{-1} = p^{-1}$, and conclude that

$$(5.16) \quad \|I_2\|_p \leq C_p t^{-5/4-1/(2\sigma_1)} \|r\|_{H^{1,1}}^2$$

for any $\sigma_1 > p$.

I_3 : We have

$$\|I_3\|_p \leq C_p t^{-3/4} \|S_k^{-1} \phi\|_{\sigma_1} \|r\|_2 \|M1\|_{\sigma_2}$$

where $\sigma_1 + \sigma_2 = 1/p$ and ϕ has support in $|k - k_c| \leq 4t^{-1/4}$. Hence for any $\sigma_1 > p$,

$$(5.17) \quad \|I_3\|_p \leq C_{p,\sigma} t^{-5/4-1/(2\sigma_1)} \|r\|_{H^{1,1}}^2.$$

I_4 : We estimate

$$\|I_4\|_p \leq C_p t^{-1} \left\| S_k^{-1} (1 - \chi) \right\|_{\sigma_1} \left\| \overline{\partial_k r} \right\|_2 \|M1\|_{\sigma_2}$$

for $\sigma_1^{-1} + \sigma_2^{-1} = p^{-1}$. Hence for any $\sigma_1 > p$,

$$(5.18) \quad \|I_4\|_p \leq C_{p,\sigma} t^{-3/2-1/(2\sigma_1)}.$$

Combining the estimates (5.15)-(5.18) give the desired estimate. \square

We can now put the pieces together.

Proof of Theorem 1.5. Write

$$\begin{aligned} \nu_1 &= 1 - t^{-1} P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r) \\ &\quad + [M^2 1 + t^{-1} P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r)] \\ &\quad + (I - M^2)^{-1} M^4 1. \end{aligned}$$

We then have

$$\nu_1 = 1 - t^{-1} P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r) + \mathcal{O}_{L^p}(t^{-1-\varepsilon})$$

for any $p > 2$, where $\mathcal{O}_{L^p}(t^{-1-\varepsilon})$ denotes a remainder bounded in the mixed space $L_k^\infty(L_z^p)$ by a constant times $t^{-1-\varepsilon}$. Recalling the reconstruction formula (5.2) we have

$$q = \mathcal{F}^{-1}(e^{itS} r) - t^{-1} \langle e^{itS} \bar{r}, P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r) \rangle + \mathcal{O}(t^{-1-\varepsilon})$$

in L_z^∞ -norm. Thus, it remains to show that

$$\langle e^{-itS} \bar{r}, P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r) \rangle = o(1)$$

in L_z^∞ -norm. Writing $\bar{r} = \bar{r}(\chi + (1 - \chi))$ we have

$$\begin{aligned} \langle e^{-itS} \bar{r}, P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r) \rangle &= \langle e^{-itS} \chi \bar{r}, P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r) \rangle \\ &\quad + \langle e^{-itS} (1 - \chi) \bar{r}, P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r) \rangle. \end{aligned}$$

By Hölder's inequality, the first right-hand term is bounded by

$$\|\chi r\|_{p'} \|P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r)\|_p$$

which is of order $t^{-1/(2p')}$ as $t \rightarrow \infty$. In the second term, we may integrate by parts to conclude that

$$|\langle e^{-itS} (1 - \chi) \bar{r}, P_k e^{-itS} \bar{r} (1 - \chi) \mathcal{S}(e^{itS} r) \rangle| \leq C t^{-1/2} \|r\|_{H^{1,1}}^3$$

\square

APPENDIX A. MULTILINEAR ESTIMATES BY MICHAEL CHRIST

In this appendix we establish a rather general multilinear inequality in terms of weak type Lebesgue spaces, then specialize it to deduce the inequality of Brown [13] stated in Proposition 2.3.

Let \mathbb{F} be one of the two fields $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, equipped with Lebesgue measure in either case. Consider \mathbb{C} -valued multilinear functionals

$$(A.1) \quad \Lambda(f_1, f_2, \dots, f_m) = \int_{\mathbb{F}^N} \prod_{j=1}^m f_j(\ell_j(y)) dy$$

where each $\ell_j : \mathbb{F}^N \rightarrow \mathbb{F}^{N_j}$ is a surjective \mathbb{F} -linear transformation, $f_j : \mathbb{F}^{N_j} \rightarrow \mathbb{C}$, and dy denotes Lebesgue measure on \mathbb{F}^N . A complete characterization of those exponents $(p_1, \dots, p_m) \in [1, \infty]^m$ for which there are inequalities of the form

$$(A.2) \quad |\Lambda(f_1, f_2, \dots, f_m)| \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}}$$

has been obtained in [11]. Such an inequality implicitly includes the assertion that the integral (A.2) converges absolutely whenever each f_j belongs to L^{p_j} . To review this result, we first recall key definitions from [10],[11].

Denote by $\dim_{\mathbb{F}}(V)$ the dimension of a vector space V over \mathbb{F} . Throughout the discussion, \mathbb{F} should be considered as fixed; vector spaces, subspaces, and linear mappings are defined with respect to \mathbb{F} .

Definition A.1. *Relative to a set of exponents $\{p_j\}$, a subspace $V \subset \mathbb{F}^N$ is said to be critical if*

$$(A.3) \quad \dim_{\mathbb{F}}(V) = \sum_j p_j^{-1} \dim_{\mathbb{F}}(\ell_j(V)),$$

to be supercritical if the right-hand side is strictly less than $\dim_{\mathbb{F}}(V)$, and to be subcritical if the right-hand side is strictly greater than $\dim_{\mathbb{F}}(V)$.

Throughout the discussion, the reciprocal of any infinite exponent is interpreted as 0. The subspace $\{0\}$ is always critical.

Theorem A.2. [11] *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $N \geq 1$ and $N_j \geq 1$ for all $j \in [1, 2, \dots, m]$. For each index $j \in [1, 2, \dots, m]$ let $\ell_j : \mathbb{F}^N \rightarrow \mathbb{F}^{N_j}$ be an \mathbb{F} -linear surjective mapping. Let $p_j \in [1, \infty]$. Then (A.2) holds if and only if \mathbb{F}^N is critical relative to $\{p_j\}$ and no proper subspace of \mathbb{F}^N is supercritical relative to $\{p_j\}$.*

This theorem was stated in [11] only for $\mathbb{F} = \mathbb{R}$, but the proof given in [11] applies equally well to $\mathbb{F} = \mathbb{C}$. See also [10] for a different proof and more thorough analysis for the case $\mathbb{F} = \mathbb{R}$.

In order to extend this theorem to include Brown's inequality (2.5), we will utilize the Lorentz spaces $L^{p,r}$ as defined for instance in [26]. These spaces are defined for $(p, r) \in [1, \infty) \times [1, \infty]$, and are Banach spaces except in the exceptional case $(p, r) = (1, 1)$. Throughout the following discussion, we assume that (p, r) is not equal to $(1, 1)$. The facts needed about the Lorentz spaces for our discussion are these:

- (i) $L^{p,p}$ equals the Lebesgue space L^p .
- (ii) $L^{p,\infty}$ equals weak L^p . That is, $f \in L^{p,\infty}(\mathbb{F}^n)$ if and only if there exists $C_f < \infty$ such that for every $\alpha \in (0, \infty)$, $|\{x \in \mathbb{F}^n : |f(x)| > \alpha\}| \leq C_f^p \alpha^{-p}$. Here $|E|$ denotes the Lebesgue measure of a subset E of \mathbb{F}^n . The infimum of all such C_f is denoted by $\|f\|_{L^{p,\infty}}$. This quantity is not in general a norm, but is equivalent to one unless $(p, r) = (1, 1)$; see [26].
- (iii) In particular, the functions $|x|^{-d/p}$ and $|z|^{-2d/p}$ belong to $L^{p,\infty}(\mathbb{R}^d)$ and to $L^{p,\infty}(\mathbb{C}^d)$, respectively.

The next result extends Theorem A.2 to Lorentz spaces, although perhaps not in the most definitive manner.

Theorem A.3. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $N \geq 1$ and $N_j \geq 1$ for all $j \in [1, 2, \dots, m]$. For each index $j \in [1, 2, \dots, m]$ let $\ell_j : \mathbb{F}^N \rightarrow \mathbb{F}^{N_j}$ be an \mathbb{F} -linear surjective mapping. Let each exponent p_j belong to the open interval $(1, \infty)$.*

Suppose that with respect to $\{p_j\}$, the total space \mathbb{F}^N is critical, and every nonzero proper subspace of \mathbb{F}^N is subcritical. Then for all exponents $r_j \in [1, \infty]$ satisfying

$$(A.4) \quad \sum_j r_j^{-1} = 1,$$

$\prod_{j=1}^m f_j \circ \ell_j$ belongs to $L^1(\mathbb{F}^N)$. Moreover, there exists $C < \infty$ independent of $\{f_j\}$ such that

$$(A.5) \quad |\Lambda(f_1, f_2, \dots, f_m)| \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j, r_j}}.$$

The proof will utilize the following crude multilinear interpolation theorem, established in [17].

Proposition A.4. *Let $a_j \in [0, \infty)$, and suppose that at least one of these numbers is nonzero. Let $\Omega = \{(t_1, \dots, t_j) \in (0, 1)^m : \sum_j a_j t_j = 1\}$, equipped with the topology induced by its embedding in $(0, 1)^m$. Let (X, \mathcal{A}, μ) be any measure space. Let $\Lambda = \Lambda(f_1, \dots, f_m)$ be a complex-valued multilinear form defined for all m -tuples of measurable simple functions $f_j : X \rightarrow \mathbb{C}$.*

Let \mathcal{O} be a nonempty open subset of Ω . Suppose that for each $t = (t_1, \dots, t_m) \in \mathcal{O}$ there exists $C_t < \infty$ such that

$$(A.6) \quad |\Lambda(f_1, \dots, f_m)| \leq C_t \prod_j \|f_j\|_{L^{p_j, 1}} \quad \text{where } p_j = t_j^{-1}$$

for all m -tuples of simple functions f_j . Then for any relatively compact subset $\mathcal{O}' \subset \mathcal{O}$ there exists $C < \infty$ such that for all $t \in \mathcal{O}'$ and all exponents r_j satisfying $\sum_{j=1}^m r_j^{-1} = 1$, for all m -tuples of measurable simple functions,

$$(A.7) \quad |\Lambda(f_1, \dots, f_m)| \leq C \prod_j \|f_j\|_{L^{p_j, r_j}} \quad \text{where } p_j = t_j^{-1}.$$

Proof of Theorem A.3. It suffices to apply Theorem A.2 and Proposition A.4 in combination. Indeed, if an m -tuple $p = \{p_j : 1 \leq j \leq m\}$ satisfies the hypotheses of Theorem A.3, then so does any m -tuple $q = \{q_j : 1 \leq j \leq m\}$ satisfying the

equation $\sum_j \frac{N_j}{N} q_j^{-1} = 1$ such that each q_j^{-1} is sufficiently close to p_j^{-1} . Indeed, as V varies over all nonzero proper subspaces of \mathbb{F}^N , the numbers $\sum_j p_j^{-1} \frac{\dim_{\mathbb{F}}(\ell_j(V))}{\dim_{\mathbb{F}}(V)}$ take on finitely many values, and are all strictly greater than one by the subcriticality hypothesis. Therefore these strict inequalities continue to hold whenever q is sufficiently close to p . The hypotheses of Proposition A.4 are thus satisfied. Applying that Proposition yields inequality (A.5). \square

Consider now the multilinear inequality of Brown [13]. Let

$$\Lambda_n(\rho, q_0, q_1, \dots, q_{2n}) = \int_{\mathbb{C}^{2n+1}} \frac{|\rho(\zeta)| |q_0(z_0)| \dots |q_{2n}(z_{2n})|}{\prod_{j=1}^{2n} |z_{j-1} - z_j|} d\mu(z)$$

where $d\mu(z)$ is product measure on \mathbb{C}^{2n+1} and $\zeta = \sum_{j=0}^{2n} (-1)^j z_j$. The inequality states that

$$(A.8) \quad |\Lambda_n(\rho, q_0, q_1, \dots, q_{2n})| \leq C \|\rho\|_2 \prod_{j=0}^{2n} \|q_j\|_2.$$

Note that since Λ_n is multilinear, it follows directly from this statement that the map

$$(\rho, q_0, q_1, \dots, q_{2n}) \mapsto \Lambda_n(\rho, q_0, q_1, \dots, q_{2n})$$

is Lipschitz continuous from any bounded subset of $(L^2(\mathbb{R}^2))^{2n+1}$ to \mathbb{C} .

To deduce (A.8) from Theorem A.3, set $\mathbb{F} = \mathbb{C}$, $N = 2n + 1$, and $m = 4n + 2$. Let the index j range over $[0, 4n + 1]$, set $N_j = 1$ for all $j \in [0, \dots, 4n + 1]$, write $z = (z_0, \dots, z_{2n})$, and consider the linear functionals $l_j : \mathbb{C}^{2n+1} \rightarrow \mathbb{C}^1$ defined by

$$(A.9) \quad l_j(z) = \begin{cases} z_j & \text{for } 0 \leq j \leq 2n, \\ z_{j-2n} - z_{j-2n-1} & \text{for } 2n < j \leq 4n, \\ \sum_{i=0}^{2n} (-1)^i z_i & \text{for } j = 4n + 1. \end{cases}$$

The following linear algebraic fact will be proved below.

Lemma A.5. *The $(4n + 2)$ -tuple of exponents $p = (p_j) = (2, 2, \dots, 2)$ satisfies the hypotheses of Theorem A.3.*

To apply the lemma to inequality (A.8), for each $j \in (2n, 4n]$ define $f_j : \mathbb{C}^1 \rightarrow \mathbb{R}^+$ by $f_j(w) = |w|^{-1}$. Each of these functions belongs to $L^{2,\infty}(\mathbb{C}^1)$. The factors $|z_j - z_{j-1}|^{-1}$ appearing in (A.8) are then $|z_j - z_{j-1}|^{-1} = f_j(\ell_j(z))$, for $j \in (2n, 4n]$. Setting $f_j = q_j$ for all $j \in [0, 2n]$ and $f_{4n+1} = \rho$, $\Lambda_n(\rho, q_0, \dots, q_{2n})$ equals $\Lambda(f_0, \dots, f_{4n+1}) = \int_{\mathbb{C}^N} \prod_{j=0}^{4n+1} f_j(\ell_j(z)) dz$. Inequality (A.8) therefore follows from Theorem A.3 together with Lemma A.5. \square

This reasoning yields various refinements of (A.8). For instance, any one of the functions ρ, q_j may be taken to be in $L^{2,\infty}(\mathbb{C})$ rather than in L^2 .

Proof of Lemma A.5. Firstly, $N = 2n + 1$, while

$$\sum_{j=0}^{4n+1} p_j^{-1} \dim_{\mathbb{C}}(\ell_j(\mathbb{C}^N)) = \sum_{j=0}^{4n+1} p_j^{-1} N_j = \sum_{j=0}^{4n+1} 2^{-1} \cdot 1 = 2^{-1} \cdot (4n + 2) = N.$$

Thus \mathbb{F}^N is critical relative to $(2, 2, \dots, 2)$.

It remains to show that any nonzero proper complex subspace V of \mathbb{C}^N is subcritical. For any index j , since ℓ_j is a linear mapping from \mathbb{C}^N to \mathbb{C}^1 , either $\dim_{\mathbb{C}}(\ell_j(V)) = 1$, or ℓ_j vanishes identically on V . Let S be the set of all $j \in [0, \dots, 2n]$ such that $z_j \equiv 0$ for all $z = (z_0, \dots, z_{2n}) \in V$, and let T be the set of all $j \in [1, 2n]$ such that $z_j - z_{j-1} \equiv 0$ for all $z \in V$, but neither j nor $j-1$ belongs to S .

The mapping $\ell_{j+2n} : V \rightarrow \mathbb{C}$ is surjective if $j \in [0, 2n]$ and $j \notin T \cup S$. For if not, then it vanishes identically; $z_j - z_{j-1} = 0$ for all $z \in V$. Since $j \notin T$, the definition of T forces at least one of the indices $j, j-1$ to belong to S , that is, at least one of the functions $z \mapsto z_j$ and $z \mapsto z_{j-1}$ vanishes identically on S . The equation $z_j - z_{j-1} \equiv 0$ then forces both of these functions to vanish identically. Therefore both indices $j, j-1$ belong to S , contradicting the hypothesis that $j \notin T \cup S$.

A further consequence is that the number of $j \in (2n, 4n]$ such that $j \notin T$, but $z_j - z_{j-1} \equiv 0$ for all $z \in V$, is at most $|S| - 1$. Equality occurs if and only if $S = [k, k-1 + |S|]$ for some $k \in [0, 2n]$.

The set of mappings $\{\ell_j : j \in S \cup T\}$ is linearly independent, and V is contained in the intersection of their nullspaces, so $\dim_{\mathbb{C}}(V) \leq 2n + 1 - |S| - |T|$. On the other hand,

$$\begin{aligned} & \sum_{j=0}^{4n+1} 2^{-1} \dim_{\mathbb{C}}(\ell_j(V)) \\ &= \sum_{j=0}^{2n} 2^{-1} \dim_{\mathbb{C}}(\ell_j(V)) + \sum_{j=2n+1}^{4n} 2^{-1} \dim_{\mathbb{C}}(\ell_j(V)) + 2^{-1} \dim_{\mathbb{C}}(\ell_{4n+1}(V)) \\ &\geq 2^{-1}(2n + 1 - |S|) + 2^{-1}(2n - |T| - (|S| - 1)) + 2^{-1} \dim_{\mathbb{C}}(\ell_{4n+1}(V)) \\ &= (2n + 1 - |S| - |T|) + 2^{-1}|T| + 2^{-1} \dim_{\mathbb{C}}(\ell_{4n+1}(V)) \\ &\geq \dim_{\mathbb{C}}(V) + 2^{-1}|T| + 2^{-1} \dim_{\mathbb{C}}(\ell_{4n+1}(V)). \end{aligned}$$

This is strictly greater than $\dim_{\mathbb{C}}(V)$ unless $T = \emptyset$, V is contained in the nullspace of ℓ_{4n+1} , $\dim_{\mathbb{C}}(V) = 2n + 1 - |S|$, and $S = [k, k-1 + |S|]$ for some $k \in [0, 2n]$ with $k-1 + |S| \leq 2n$.

Suppose that $T = \emptyset$, and that V is contained in the nullspace of ℓ_{4n+1} . S cannot be all of $[0, 2n]$, for this would force $V = \{0\}$, contrary to hypothesis. Therefore the equation $\ell_{4n+1}|_V \equiv 0$ is not forced by the equations $\ell_j|_V \equiv 0$ for all $j \in S$, so $\dim_{\mathbb{C}}(V)$ must be strictly less than $2n + 1 - |S|$. Therefore $\sum_{j=0}^{4n+1} 2^{-1} \dim_{\mathbb{C}}(\ell_j(V))$ is strictly greater than $\dim_{\mathbb{C}}(V)$ in all cases; every nonzero proper subspace of \mathbb{C}^N is subcritical. \square

APPENDIX B. TIME EVOLUTION OF SCATTERING MAPS

The purpose of this appendix is to give a self-contained proof that the function q defined by (1.9) solves the DS II equation for $q_0 \in \mathcal{S}(\mathbb{R}^2)$. Previous proofs may be found, for example, in the papers of Beals-Coifman [7, 8, 9] and Sung [28], Part III. We suppose that $r \in C^1(\mathbb{R}_t; \mathcal{S}(\mathbb{R}^2))$ obeys a linear equation

$$\dot{r} = i\varphi r$$

where φ is a real-valued polynomial in k and \bar{k} . We will obtain an effective formula for \dot{q} if $q = \mathcal{I}(r)$ by differentiating

$$q = \langle e_k \bar{r}, \nu_1 \rangle$$

and exploiting solutions $(\nu_1^\#, \nu_2^\#)$ to a ‘dual’ problem

$$(B.1) \quad \begin{aligned} \bar{\partial}_k \nu_1^\# &= \frac{1}{2} e_k \overline{r^\# \nu_2^\#} \\ \bar{\partial}_k \nu_2^\# &= \frac{1}{2} e_k \overline{r^\# \nu_1^\#} \end{aligned}$$

where $r^\# = \bar{r}$. The following lemma on symmetries of the map \mathcal{R} shows that $r^\# = \mathcal{R}(q^\#)$ where $q^\#(z) = \overline{q(-z)}$.

Lemma B.1. *Let $q, q^\flat \in H^{1,1}$ and let $r = \mathcal{R}(q)$, $r^\flat = \mathcal{R}(q^\flat)$.*

- (i) *If $q^\flat(z) = -q(z)$, then $r^\flat(k) = -r(k)$,*
- (ii) *if $q^\flat(z) = -q(-z)$, then $r^\flat(k) = -r(-k)$, and*
- (iii) *if $q^\flat(z) = \overline{q(z)}$, then $r^\flat(k) = \overline{-r(k)}$.*

Proof. In what follows we let $(\mu_1^\flat, \mu_2^\flat)$ denote the solutions to (1.11) with q replaced by q^\flat .

- (i) follows from (1.4) and the fact that $\mu_1^\flat = \mu_1$.
- (ii) follows from (1.4) and the fact that $\mu_1^\flat(z, k) = \mu_1(-z, -k)$
- (iii) From the definition (1.4) we compute (recall (2.1))

$$\begin{aligned} r^\flat(k) &= \langle e_{-k} \bar{q}, \mu_1^\flat \rangle \\ &= \langle e_{-k} q, (I - \bar{P}_k e_{-k} q P_k e_k \bar{q})^{-1} 1 \rangle \\ &= \langle (I - e_{-k} q \bar{P}_k \bar{q} e_k P_k)^{-1} e_{-k} q, 1 \rangle \\ &= \langle (I - \bar{P}_k \bar{q} e_k P_k e_{-k} q)^{-1} 1, e_k \bar{q} \rangle \\ &= \overline{r(-k)} \end{aligned}$$

as claimed. □

From the formula

$$[\partial_t, T_k^2] = -\frac{i}{4} P_k e_k \bar{r} [\varphi, \bar{P}_k] e_{-k} r$$

we have

$$\begin{aligned} \dot{\nu}_1 &= \left[\partial_t, (I - T_k^2)^{-1} \right] 1 \\ &= -\frac{i}{4} (I - T_k^2)^{-1} P_k e_k \bar{r} [\varphi, \bar{P}_k] e_{-k} r \nu_1 \end{aligned}$$

so that

$$\begin{aligned} \dot{q} &= i \langle e_k \varphi \bar{r}, \nu_1 \rangle - \frac{i}{4} \langle e_k \bar{r}, (I - T_k^2)^{-1} P_k e_k \bar{r} [\varphi, \bar{P}_k] e_{-k} r \nu_1 \rangle \\ &= i \langle e_{-k} r f_1, \varphi g_1 \rangle + i \langle f_2, \varphi e_{-k} r \nu_1 \rangle \end{aligned}$$

where

$$\begin{aligned} f_1 &= \overline{P}_k \left(I - (T_k^2)^* \right)^{-1} e_k \overline{r}, \\ g_1 &= \overline{P}_k e_{-k} r \nu_1, \\ f_2 &= 1 + e_{-k} r \overline{P}_k \left(I - (T_k^2)^* \right)^{-1} e_k \overline{r}. \end{aligned}$$

Noting that $(T_k^2)^* = \frac{1}{4} e_k \overline{r} \overline{P}_k e_{-k} r P$, it is not difficult to see that

$$\begin{aligned} f_1(z, k) &= \overline{\nu_2^\#(-z, k)}, \\ g_1(z, k) &= \overline{\nu_2(z, k)}, \\ f_2(z, k) &= \nu_1^\#(-z, k), \end{aligned}$$

so that

$$\dot{q}(z, t) = i \left\langle e_{-k} r \overline{\nu_2^\#(-z, \cdot)}, \varphi \overline{\nu_2(z, \cdot)} \right\rangle + i \left\langle \nu_1^\#(-z, \cdot), \varphi e_{-k} r \nu_1(z, \cdot) \right\rangle$$

where we have suppressed the t -dependence of ν and $\nu^\#$. Setting

$$\eta(z, k) = \frac{1}{2} e_k(z) \overline{r}(k) \nu_2^\#(-z, k) \overline{\nu_2(z, k)} + \frac{1}{2} \overline{\nu_1^\#(-z, k)} e_{-k}(z) r(k) \nu_1(z, k)$$

we have

$$\dot{q}(z) = 2i \int \varphi(k) \eta(z, k) \, dm(k).$$

Using (1.12) and (B.1), we can write

$$\eta(z, k) = \overline{\partial}_k \left[\nu_2^\#(-z, k) \nu_1(z, k) \right]$$

and

$$\overline{\eta(z, k)} = \overline{\partial}_k \left[\nu_1^\#(-z, k) \nu_2(z, k) \right]$$

so that, if $\varphi(k) = 4 \operatorname{Re}(k^2)$, we conclude that

$$\dot{q}(z) = 4i (I_1 + \overline{I_2})$$

(the complex conjugate on I_2 is intentional) where

$$\begin{aligned} I_1 &= \int k^2 \overline{\partial}_k \left[\nu_2^\#(-z, k) \nu_1(z, k) \right] \, dm(k), \\ I_2 &= \int k^2 \overline{\partial}_k \left[\nu_1^\#(-z, k) \nu_2(z, k) \right] \, dm(k). \end{aligned}$$

The integrands in I_1 and I_2 are exact differentials and, for $r \in \mathcal{S}(\mathbb{R}^2)$, vanish rapidly at infinity. We can evaluate I_1 and I_2 using the fact that, if h is a smooth function with $\overline{\partial}h$ of rapid decay and

$$(B.2) \quad h \sim \sum_{j \geq 0} \frac{h_j}{k^{j+1}}$$

then

$$\int k^n \overline{\partial}_k h \, dm(k) = 2\pi i h_n.$$

We compute the large- k asymptotic expansions of ν_1 and ν_2 in Appendix C. Write $[h]_j$ for h_j in the expansion (B.2). In terms of the expansion (C.2) we have

$$\begin{aligned}\left[\nu_2^\#(-z, k)\nu_1(z, k)\right]_2 &= \nu_{2,0}^\#\nu_{1,2} + \nu_{2,1}^\#\nu_{1,1} + \nu_{2,2}^\#\nu_{1,0}, \\ \left[\nu_1^\#(-z, k)\nu_2(z, k)\right]_2 &= \nu_{2,0}^\#\nu_{1,2} + \nu_{2,1}^\#\nu_{1,1} + \nu_{2,2}^\#\nu_{1,0}\end{aligned}$$

where $\nu^\#$ corresponds to the potential $q^\#$, and, since $\nu^\#$ is evaluated at $-z$, we replace q by $-\bar{q}$, P by $-P$, and ∂ by $-\partial$ in (C.3) and (C.4)-(C.6) to find the expansion coefficients for $\nu^\#$. Straightforward computation using (C.3) and (C.4)-(C.6) gives (recall (2.3))

$$\left[\nu_2^\#(-z, k)\nu_1(z, k)\right]_2 = \frac{1}{4}q\left(\mathcal{S}\left(|q|^2\right)\right) - \frac{1}{2}\partial^2q$$

where we used the identity $\left(\bar{\partial}^{-1}f\right)^2 = 2\bar{\partial}^{-1}\left(f\bar{\partial}^{-1}f\right)$ with $f = |q|^2$ to eliminate terms of fifth order in q . Similarly,

$$\left[\nu_1^\#(-z, k)\nu_2(z, k)\right]_2 = -\frac{1}{4}q\left(\mathcal{S}\left(|q|^2\right)\right) + \frac{1}{2}\partial^2\bar{q}.$$

Finally, we obtain

$$i\dot{q}(z) = -2(\partial^2q + \bar{\partial}^2q) - q(g + \bar{g})$$

where

$$g = -\mathcal{S}\left(|q|^2\right).$$

This is exactly the DS II equation.

APPENDIX C. ASYMPTOTIC EXPANSIONS

In this section we compute large-parameter asymptotic expansions of the solutions $\nu = (\nu_1, \nu_2)$ of (1.12). Exploiting the fact that $\nu = (\mu_1, e_k\bar{\mu}_2)$, we conclude from (1.11) that

$$\begin{aligned}\text{(C.1)} \quad \bar{\partial}_z\nu_1 &= \frac{1}{2}q\nu_2 \\ (\partial_z + k)\nu_2 &= \frac{1}{2}\bar{q}\nu_1\end{aligned}$$

For $r \in \mathcal{S}(\mathbb{R}^2)$, the functions (ν_1, ν_2) admit a large- k asymptotic expansion of the form

$$\text{(C.2)} \quad \nu \sim (1, 0) + \sum_{\ell \geq 0} k^{-(\ell+1)}\nu^{(\ell)}$$

where $\nu^{(\ell)} = (\nu_{1,\ell}, \nu_{2,\ell})^T$. From the system (C.1) we easily deduce that

$$\text{(C.3)} \quad \nu_{1,0} = \frac{1}{4}\bar{\partial}^{-1}\left(|q|^2\right), \quad \nu_{2,0} = \frac{1}{2}\bar{q}$$

while for $\ell \geq 1$,

$$\begin{aligned}\nu_{2,\ell} &= \frac{1}{2}\bar{q}\nu_{1,\ell-1} - \partial\nu_{2,\ell-1} \\ \nu_{1,\ell} &= \frac{1}{2}P(q\nu_{2,\ell}).\end{aligned}$$

It easily follows that

$$(C.4) \quad \nu_{1,1} = \frac{1}{16}P\left(|q|^2 P\left(|q|^2\right)\right) - \frac{1}{4}P(q\partial\bar{q}),$$

$$(C.5) \quad \nu_{2,1} = \frac{1}{8}\bar{q}P\left(|q|^2\right) - \frac{1}{2}\partial\bar{q},$$

$$(C.6) \quad \begin{aligned}\nu_{2,2} &= \frac{1}{32}\bar{q}P\left(|q|^2 P\left(|q|^2\right)\right) \\ &\quad - \frac{1}{8}\partial\left(\bar{q}P\left(|q|^2\right)\right) - \frac{1}{8}\bar{q}P(q\partial\bar{q}) + \frac{1}{2}\partial^2\bar{q}.\end{aligned}$$

Remark C.1. In a similar way one can show that for $r \in \mathcal{S}(\mathbb{R}^2)$, μ has a large- z asymptotic expansion whose coefficients are computed in terms of r and its derivatives. Thus for example

$$\begin{aligned}\mu_1(z, k) &= 1 + \frac{1}{z}\left(\frac{1}{4}\bar{\partial}_k^{-1}\left(|r|^2\right)\right) + \mathcal{O}\left(|z|^{-2}\right), \\ \mu_2(z, k) &= \frac{1}{z}\left(\frac{1}{2}r\right) + \mathcal{O}\left(|z|^{-2}\right).\end{aligned}$$

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